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A Lie algebraic approach to effective mass Schrödinger equations

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Abstract

We use Lie algebraic techniques to obtain exact solutions of the effective mass Schrödinger equation. In particular we use the $su(1,1)$ algebra, both as a spectrum generating algebra and as a potential algebra, to obtain exact solutions of effective mass Schrödinger equations corresponding to a number of potentials. We also discuss the construction of isospectral Hamiltonians for which both the mass and the potential are different.

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1. Introduction

Over the years the Schrödinger equation has been studied extensively regarding its exact solvability. Many advances have been made in this area by classifying quantum mechanical potentials according to their symmetry properties, For instance, various algebras which reveal the underlying symmetry as well as facilitating obtaining the solutions have been found. Generally the symmetry algebras are of two types—spectrum generating algebras (SGAs) [1] and potential algebras (PAs) [2]. Using SGAs it is possible to determine the entire spectrum of a particular potential while in the case of PAs one can obtain a family of potentials having the same energy.

In contrast to the standard Schrödinger equation, the study of the Schrödinger equation with a position-dependent effective mass has been the subject of recent interest. Such quantum systems have been found to be useful in the study of electronic properties of semiconductors [3], quantum dots [4], liquid crystals [5] etc. Although exact solutions are difficult to obtain some exactly soluble models of effective mass Schrödinger equations have been found [6–8]. Supersymmetric techniques have also been used in obtaining exact solutions [9, 10]. On the other hand, Lie algebraic methods have been widely used not only in quantum mechanics [1] but in other areas too [11, 12]. However, such methods have so far not been used to analyse effective mass Schrödinger equations. In this article our aim is to examine such equations from the point of view of their Lie algebraic symmetry. In particular we shall use the $su(1,1)$

algebra, both as an SGA and as a PA to obtain the spectrum of different types of potential. The organization of the paper is as follows: in section 2 we describe the application of the $su(1,1)$ algebra as an SGA to obtain the spectrum of potentials which exhibit the harmonic oscillator and the singular oscillator spectrum; in section 3 we use the $su(1,1)$ algebra as a PA to obtain the spectra of Morse-like and solition-like potentials; in section 4 we discuss the construction of isospectral Hamiltonians using the $su(1,1)$ algebra; finally, section 5 is devoted to a conclusion.

2. $su(1,1)$ algebra as a spectrum generating algebra

The application of $su(1,1) \sim so(2,1)$ algebra as an SGA has a long history. Different realizations have been used by various authors to obtain the spectrum of different potentials. However, before we begin let us first present a brief account of the Schrödinger equation in the effective mass approximation. We note that there are several ways to define the kinetic energy when the mass depends on position. A general form of the kinetic energy operator is given by [13]

$$T = \frac{1}{4}(m^{\eta}\boldsymbol{p}m^{\epsilon}\boldsymbol{p}m^{\rho} + m^{\rho}\boldsymbol{p}m^{\epsilon}\boldsymbol{p}m^{\eta}) \tag{1}$$

with $\eta + \epsilon + \rho = -1$. Depending on the choice of the parameters different forms of kinetic energy operators emerge:

$$T = \frac{1}{4}\left[\frac{1}{m}p^2 + p^2\frac{1}{m}\right] \tag{2}$$

$$T = \frac{1}{2}\left[\frac{1}{\sqrt{m}}p^2\frac{1}{\sqrt{m}}\right] \tag{3}$$

$$T = \frac{1}{2}\left[\boldsymbol{p}\frac{1}{m}\boldsymbol{p}\right]. \tag{4}$$

Here we shall be following Lévy-Leblond [14] and in this case the kinetic energy operator is given by (4). The corresponding Schrödinger equation reads

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2m(x)}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right) + (E - V(x))\psi(x) = 0. \tag{5}$$

The wavefunction $\psi(x)$ has to be continuous across the abrupt interface and its derivative should satisfy the condition

$$\left.\frac{1}{m(x)}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right|_{-} = \left.\frac{1}{m(x)}\frac{\mathrm{d}\psi(x)}{\mathrm{d}x}\right|_{+}. \tag{6}$$

We now proceed to obtain the spectrum of (5) correponding to various potentials. As mentioned before there are different realizations of the $su(1,1)$ algebra which are appropriate for studying different types of problem. For example one may use a realization in which two of the generators consist of a second-order differential operator only and some scalar function [15, 16]. Now looking at (5) we find that the Schrödinger equation contains not only a second-order derivative term but it also a first-order derivative term. Therefore in the present case it would be advantageous to use a realization of the $su(1,1)$ generators containing both second-order and first-order differential operators of a single variable. Thus we consider generators Γ_i, $i = 1, 2, 3$, of the form [17]

$$\begin{aligned} \Gamma_1 &= u^2(x)\,\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x)\frac{\mathrm{d}}{\mathrm{d}x} + w(x) + \frac{\phi^2(x)}{16} \\ \Gamma_2 &= -\frac{\mathrm{i}}{2}\phi(x)\left(u(x)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}p(x)\right) - \frac{\mathrm{i}}{4} \\ \Gamma_3 &= u^2(x)\,\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x)\frac{\mathrm{d}}{\mathrm{d}x} + w(x) - \frac{\phi^2(x)}{16} \end{aligned} \tag{7}$$

where

$$\begin{aligned} u(x) &= \frac{1}{\phi'(x)} \\ p(x) &= \frac{v(x)}{u(x)} - u'(x) \\ w(x) &= \frac{p^2(x)}{4} + \frac{u(x)p'(x)}{2} - \frac{g^2}{\phi^2(x)}. \end{aligned} \tag{8}$$

In (7) and (8) $u(x)$ (or $\phi(x)$) and $v(x)$ (or $p(x)$) are arbitrary differentiable functions and g^2 is an arbitrary constant (which will be shown later to be related to the eigenvalues of the Casimir operator). It can be verified that the generators Γ_i, $i = 1, 2, 3$, satisfy the $su(1, 1)$ commutation relations:

$$[\Gamma_1, \Gamma_2] = -\mathrm{i}\Gamma_3, \qquad [\Gamma_2, \Gamma_3] = \mathrm{i}\Gamma_1, \qquad [\Gamma_3, \Gamma_1] = \mathrm{i}\Gamma_2. \tag{9}$$

The above form of the generators is canonical in the sense that they are form invariant with respect to a variable transformation [17].

Now unitary irreducible representations (UIRs) of the $su(1, 1)$ Lie algebra in which the compact generator Γ_3 is diagonal can be classified according to the eigenvalues of the Casimir operator $C = (\Gamma_3^2 - \Gamma_2^2 - \Gamma_1^2)$ and the compact generator Γ_3. Let us now denote the eigenvalues of the Casimir operator C and the compact generator Γ_3 by q and N respectively. Then it can be shown that [15]

$$q = -\frac{3}{16} + \frac{g^2}{4} = j(j+1), \qquad j = j_\pm = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{\tfrac{1}{4} + g^2} \tag{10}$$

$$N = E_0 + n \tag{11}$$

where E_0 is a real number and n is an integer.

The UIR of the $su(1, 1)$ algebra can be classified into the following categories:

(1) continuous principal series $d_p(j)$
(2) continuous supplementary series $d_s(j)$
(3) discrete series $D^+(j)$ and
(4) discrete series $D^-(j)$.

Out of these four categories the first two are related to the continuous spectrum while the latter two are related to the discrete spectrum. For the $D^-(j)$ representation j is real and $j < 0$, and $E_0 = j$ so $N = (n + j)$. Thus in this case the spectrum of Γ_3 is bounded from above. However, for the $D^+(j)$ representation j is a real negative number and $E_0 = -j$ so that $N = n - j = -j, -j + 1, -j + 2, \ldots$. Thus in this case the spectrum of Γ_3 is bounded from below. Since in this section we shall consider quantum systems with infinite discrete spectrum bounded from below we shall require only the representation $D^+(j)$.

Let us now consider the operator

$$\Gamma = u^2(x)\,\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v(x)\frac{\mathrm{d}}{\mathrm{d}x} + w(x) - b\phi^2(x) = \left[\left(\frac{1}{2} - 8b\right)\Gamma_1 + \left(\frac{1}{2} + 8b\right)\Gamma_3\right], \qquad b \neq 0. \tag{12}$$

Then using the transformations [15]

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\theta\Gamma_2}\Gamma_1\mathrm{e}^{\mathrm{i}\theta\Gamma_2} &= \Gamma_1\cosh\theta + \Gamma_3\sinh\theta \\ \mathrm{e}^{-\mathrm{i}\theta\Gamma_2}\Gamma_3\mathrm{e}^{\mathrm{i}\theta\Gamma_2} &= \Gamma_1\sinh\theta + \Gamma_3\cosh\theta \end{aligned} \tag{13}$$

the eigenvalue equation for Γ (which is essentially equation (5))

$$\Gamma\psi(x) = \mu\psi(x) \tag{14}$$

can be transformed to the eigenvalue equation for the compact generator Γ_3:

$$\Gamma_3\psi(x) = \frac{\mu}{4\sqrt{b}}\psi(x). \tag{15}$$

Now from (10) it follows that only one value of j $(=j_-)$ is allowed and using (11) we obtain from (15)

$$\mu = 4\sqrt{b}(n - j_-) = 4\sqrt{b}\left(n + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{4} + g^2}\right). \tag{16}$$

Next we note that the Schrödinger operator in (5) and the operator Γ in (12) have similar forms. On identifying these two operators we obtain

$$u^2(x) = \frac{1}{2m(x)}, \qquad v(x) = -\frac{m'(x)}{2m^2(x)} \tag{17}$$

$$V(x) = b\phi^2(x) - w(x) \tag{18}$$

$$E_n = 4\sqrt{b}\left(n + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{4} + g^2}\right). \tag{19}$$

It follows therefore that if $m(x)$ is known then $u(x)$ (and hence $\phi(x)$), $v(x)$ and $w(x)$ become known through the relations (17) and (8). Thus the eigenvalues of the effective mass Schrödinger equation (5) are given by equation (19). However to obtain specific potentials it is necessary to specify the mass $m(x)$. Here we choose the mass to be the same as in [9] (we would like to mention that various other choices of the mass are possible, for instance exponentially rising mass as in [8]):

$$m(x) = \left(\frac{\alpha + x^2}{1 + x^2}\right)^2. \tag{20}$$

Then from (8), (17) and (18) the potential is found to be

$$\begin{aligned} V(x) = 2b[x + (\alpha - 1)\tan^{-1}x]^2 &+ \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[3x^4 - (2\alpha - 4)x^2 - \alpha] \\ &+ \frac{g^2}{2[x + (\alpha - 1)\tan^{-1}x]^2}. \end{aligned} \tag{21}$$

Thus in the effective mass formulation, (21) represents the singular oscillator potential (the singularity being at $x = 0$) with energy given by (19).

Let us now consider the case $g^2 = 0$. It follows from (10) that in this case both the values of j are acceptable (since for $g^2 = 0$ both $j_\pm < 0$) and thus to obtain the complete spectrum both these values have to be used. In this case the potential is given by

$$V(x) = 2b[x + (\alpha - 1)\tan^{-1}x]^2 + \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[3x^4 - (2\alpha - 4)x^2 - \alpha] \tag{22}$$

while the corresponding energy is given by

$$E_n = 2\sqrt{b}(n + \tfrac{1}{2}). \tag{23}$$

The potential (22) which is the effective mass analogue of the standard harmonic oscillator was previously obtained in [9] using the shape invariance criteria. It can also be seen that when $\alpha = 1$ the potentials (21) and (22) reduce to standard singular and harmonic oscillator potentials respectively [18].

3. $su(1,1)$ algebra as a potential algebra

PAs are distinct from SGAs in the sense that in this case we obtain a family of potentials with the same eigenvalue. This approach was initiated by Alhassid *et al* [2] and it was used widely in the case of the standard Schrödinger equation with a constant mass. Here we shall obtain a number of potentials for which the effective mass Schrödinger equation admits exact solutions.

We note that both the compact generator Γ_3 and the Casimir operator C are simultaneously diagonalizable. In the case when $su(1,1)$ acts as an SGA we identified the Schrödinger operator with the compact generator Γ_3. In the present case instead of Γ_3 we express the Hamiltonian as a linear function of the Casimir operator C:

$$H = -\tfrac{1}{4} - C \tag{24}$$

where C is given by

$$C = \Gamma_3^2 - \Gamma_2^2 - \Gamma_1^2 = \Gamma_3^2 - \tfrac{1}{2}(\Gamma_+\Gamma_- + \Gamma_-\Gamma_+) \tag{25}$$

and we have defined $\Gamma_\pm = (\Gamma_1 \pm \mathrm{i}\Gamma_2)$. In terms of $\Gamma_\pm$ and Γ_3 the commutation relations (9) read

$$[\Gamma_3, \Gamma_\pm] = \pm\Gamma_\pm, \qquad [\Gamma_+, \Gamma_-] = -2\Gamma_3. \tag{26}$$

It may be noted that if $|jN\rangle$ is a simultaneous eigenstate of C and Γ_3 then it is also an eigenstate of the Hamiltonian H with eigenvalue $-(j + \frac{1}{2})^2$. Following Sukumar [19] we now consider the following representation of the generators:

$$\begin{aligned} \Gamma_3 &= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\phi} \\ \Gamma_\pm &= \exp(\pm\mathrm{i}\phi)\left[\pm h(x)\frac{\mathrm{d}}{\mathrm{d}x} \pm g(x) + f(x)\Gamma_3 + c(x)\right] \end{aligned} \tag{27}$$

where in order that Γ_3 and $\Gamma_\pm$ satisfy the algebra (26) the functions $h(x)$, $g(x)$ and $c(x)$ should satisfy the equations

$$f^2(x) - h(x)\frac{\mathrm{d}f(x)}{\mathrm{d}x} = 1, \qquad h(x)\frac{\mathrm{d}c(x)}{\mathrm{d}x} - c(x)f(x) = 0. \tag{28}$$

The equations (28) can be easily integrated and their solutions are given by

$$\begin{aligned} f(x) &= -\tanh\int_{x_0}^{x}\frac{\mathrm{d}y}{h(y)} \\ c(x) &= A\,\mathrm{sech}\int_{x_0}^{x}\frac{\mathrm{d}y}{h(y)} \end{aligned} \tag{29}$$

where A and x_0 are constants of integration. Then using (25) and (27) the Schrödinger equation can be written as

$$\begin{aligned} H\psi = \Bigg[-\frac{1}{4} + (f^2 - 1)\Gamma_3^2 - h^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \left(h\frac{\mathrm{d}h}{\mathrm{d}x} + 2gh - fh\right)\frac{\mathrm{d}}{\mathrm{d}x} \\ + \left(fg - g^2 - h\frac{\mathrm{d}g}{\mathrm{d}x}\right) + (2cf\Gamma_3 + c^2)\Bigg]\psi = -\left(j + \frac{1}{2}\right)^2\psi. \end{aligned} \tag{30}$$

Now if we identify equations (5) and (30) then we obtain

$$h^2(x) = \frac{1}{2m(x)} \tag{31}$$

$$g(x) = \frac{1}{2}\left[\frac{\mathrm{d}h(x)}{\mathrm{d}x} + f(x)\right] \tag{32}$$

$$V(x) = \left(fg - g^2 - h\frac{\mathrm{d}g}{\mathrm{d}x}\right) + (2cf\Gamma_3 + c^2) + (f^2 - 1)\Gamma_3^2. \tag{33}$$

We now consider the mass to be the same as in (20). Then several potentials can be obtained by suitably choosing the integration constant x_0.

Case 1. Let us choose $x_0 = -\infty$. Then from (29), (31) and (32) we obtain

$$f(x) = -1 \tag{34}$$

$$c(x) = A\exp\left[\sqrt{2}(x + (\alpha - 1)\tan^{-1}x)\right] \tag{35}$$

$$g(x) = \frac{1}{\sqrt{2}}\frac{(\alpha - 1)x}{(\alpha + x^2)^2} - \frac{1}{2}. \tag{36}$$

The potential can now be obtained from (33) and is given by

$$V(x) = N^2\left\{\exp\left[-2\left(\sqrt{2}t - t_0\right)\right] - 2\exp\left[-\left(\sqrt{2}t - t_0\right)\right]\right\} + \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[3x^4 + (4 - 2\alpha)x^2 - \alpha] \tag{37}$$

where t_0 is a certain constant and we have used $t = x + (\alpha - 1)\tan^{-1}x$. The corresponding energy spectrum can be found from (30) and is given by

$$E_n = -(n - N + \tfrac{1}{2})^2, \qquad n = 0, 1, 2, \ldots, \bar{n} \leqslant N - \tfrac{1}{2}. \tag{38}$$

The potential in (37) exhibits the spectrum of the Morse potential. Now if N is kept fixed and A is allowed to vary then we obtain a family of potentials with the same eigenvalue (38). We note that the potential in (37) is exactly the same as that obtained in [9] from a supersymmetric consideration.

Case 2. Let us now choose $x_0 = 0$. Again from (29), (31) and (32) we obtain

$$f(x) = -\tanh\sqrt{2}[x + (\alpha - 1)\tan^{-1}x] \tag{39}$$

$$c(x) = A\,\mathrm{sech}\sqrt{2}[x + (\alpha - 1)\tan^{-1}x] \tag{40}$$

$$g(x) = \frac{1}{\sqrt{2}}\frac{(\alpha - 1)x}{(\alpha + x^2)^2} - \frac{1}{2}\tanh\sqrt{2}[x + (\alpha - 1)\tan^{-1}x]. \tag{41}$$

The potential in this case reads

$$V(x) = -(N^2 - A^2 - \tfrac{1}{4})\mathrm{sech}^2\left(\sqrt{2}t\right) - 2NA\,\mathrm{sech}\left(\sqrt{2}t\right)\tanh\left(\sqrt{2}t\right) + \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[3x^4 + (4 - 2\alpha)x^2 - \alpha] \tag{42}$$

and the corresponding spectrum is the same as that given before:

$$E_n = -(n - N + \tfrac{1}{2})^2, \qquad n = 0, 1, 2, \ldots, \bar{n} \leqslant N - \tfrac{1}{2}. \tag{43}$$

Now for $A = 0$ we obtain from (42)

$$V(x) = -\left(N^2 - \frac{1}{4}\right)\mathrm{sech}^2\left(\sqrt{2}t\right) + \frac{(\alpha - 1)}{2(\alpha + x^2)^4}[3x^4 + (4 - 2\alpha)x^2 - \alpha] \tag{44}$$

and the corresponding energy is the same as in (43). We note that for $\alpha = 1$ the potential (44) reduces to the well known soliton potential.

4. Construction of isospectral Hamiltonians

In the case of the constant mass Schrödinger equation there are many methods of generating isospectral Hamiltonians [20]. When the mass depends on the position one can use the tools of supersymmetric quantum mechanics (SUSYQM) to construct a pair of isospectral Hamiltonians with the same mass but different potentials [9]. In a recent paper [10] it was shown that SUSYQM can still be used to construct isospectral Hamiltonians for which both the mass and the potential are different. Here our aim is to study the same problem from the point of view of Lie algebra. To be more specific we shall use the formalism of the previous sections to construct isospectral Hamiltonians with different masses and potentials.

From the equations (17), (18), (32) and (33) it is clear that the final form of the potential depends, apart from other factors, on the form of the mass. It is important to note that irrespective of the potential the spectrum does not change if the mass is a well behaved function of the space coordinate. This is an indication that isospectral Hamiltonians can be constructed even when the mass and the potential are different. To illustrate this idea here we shall work with a mass different from (20). As an example let us consider the mass to be of the form

$$m(x) = \left(\frac{\alpha + x^2}{1 + x^2}\right)^4. \tag{45}$$

Now using (45) in (8) and (17) we find from (18)

$$\begin{aligned} V(x) = \frac{b}{2}\left[2x + \frac{(\alpha-1)^2 x}{x^2+1} + (\alpha-1)(\alpha+3)\tan^{-1}x\right]^2 \\ + (\alpha-1)(1+x^2)^2\left[\frac{3x^4 + (7-5\alpha)x^2 - \alpha}{(\alpha+x^2)^6}\right] \\ + 2g^2/\left[2x + \frac{(\alpha-1)^2 x}{x^2+1} + (\alpha-1)(\alpha+3)\tan^{-1}x\right]^2 \end{aligned} \tag{46}$$

and the corresponding energy is the same as in (19):

$$E_n = 4\sqrt{b}\left(n + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{4} + g^2}\right). \tag{47}$$

Thus the Schrödinger equations (5) with masses (20) and (45) and potentials (18) and (46) respectively have the same spectrum. In other words it has been shown that two Schrödinger equations with different masses and potentials possess the same spectrum. Note that if we take $g^2 = 0$ then we can obtain a potential which will exhibit the harmonic oscillator spectrum (23).

Let us now construct another example using $su(1,1)$ as a PA. Proceeding as in section 2 we find that for $x_0 = 0$

$$\begin{aligned} V(x) = -(N^2 - A^2 - \tfrac{1}{4})\operatorname{sech}^2(\sqrt{2}z) - 2NA\operatorname{sech}(\sqrt{2}z)\tanh(\sqrt{2}z) \\ + (\alpha-1)(1+x^2)^2\left[\frac{3x^4 + (7-5\alpha)x^2 - \alpha}{(\alpha+x^2)^6}\right] \end{aligned} \tag{48}$$

where we have defined $z = [x + (\alpha-1)^2 x/2(1+x^2) + [(\alpha-1)(\alpha+3)/2]\tan^{-1}x]$. The potential in (48) has the spectrum

$$E_n = -(n - N + \tfrac{1}{2})^2, \qquad n = 0, 1, 2, \ldots, \bar{n} \leqslant N - \tfrac{1}{2}. \tag{49}$$

Thus the potentials (42) and (48) are isospectral. It is therefore clear that we can obtain isospectral potentials with different masses.

5. Conclusion

In this article we have discussed application of the $su(1, 1)$ algebra to Schrödinger equations with an effective mass. In particular using the $su(1, 1)$ algebraic methods we have obtained the exact spectrum corresponding to a number of potentials. In this context we note that to obtain the wavefunctions one has to consider the $su(1, 1)$ algebra in the form $[\Gamma_3, \Gamma_\pm] = \pm\Gamma_\pm$, $[\Gamma_+, \Gamma_-] = -2\Gamma_3$. Then the wavefunctions $\psi_n \sim (\Gamma_+)^n\psi_0$ can be obtained from the ground state ψ_0 which satisfies the relation $\Gamma_-\psi_0 = 0$.

We note that some of the potentials obtained here were previously found using supersymmetric methods while the others are new. In a sense therefore the present study compliments [9, 10]. Another issue which we have considered here is the question of isospectrality of different Schrödinger equations with different masses. It has also been explicitly shown that two Schrödinger equations with different masses and potentials can be exactly isospectral. The reason for this is that both the potentials (21) and (46) are $su(1, 1)$ symmetric. The same is true for the potentials (42) and (48) for which $su(1, 1)$ is a PA.

Another interesting point to note is that effective mass Schrödinger equations are particularly suitable for Lie algebraic treatment (at least for the set of generators considered here). This is because the effective mass Schrödinger equations contain a first-derivative term so they can be directly identified with one of the generators. On the other hand in the case of the constant mass Schrödinger equation one has to perform a transformation on one of the generators to eliminate the first-derivative term so that it can be identified with the Schrödinger equation. In this context we note that it is possible to transform the effective mass Schrödinger equation to a constant mass Schrödinger equation and that the latter equation may be solved algebraically or otherwise. However in this case one cannot make sure that the symmetry of the transformed equation is shared by the original equation also. In other words the symmetry of the variable mass Schrödinger equation may not survive the transformation to a constant mass equation. This is why we have considered the effective mass Schrödinger equation in its original form (5). Finally we would like to mention that it would be interesting to search for more general potentials following the methods of [21, 22].

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